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CORRELATION OF CERTAIN COLLISION PROBLEMS WITH RADIATION THEORY

BY

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Introduction.

It is possible in certain collision problems, involving charged particles, to make a correlation with radiation theory by analysing the perturbing fields of the particles into pure harmonic waves, and comparing the effect of these components with that of homogeneous radiation of corresponding intensity and frequency. Such a procedure was used by FERMI[1] (1924) to calculate the ionisation of atoms by α-particles. More recently its application to radiative collisions has been considered by WEISZÄCKER[2] and the writer.[3] In this paper a further general development of the method is made, with applications to the production of pairs of positive and negative electrons by high-energy photons and electrons, the splitting of a photon by an electric particle, radiative collisions, nuclear disintegration by electrons, and other problems. In some of these problems new results are arrived at, and in others a check obtained on previous results. The value of the method, however, lies perhaps not so much in this as in the new view-point provided by an analysis of the problems dealt with into two parts viz.

[1] Zs. f. Phys. 29, 315, 1924.

[2] Zs. f. phys. 88, 612, 1934.

[3] Phys. Rev. 45, 729, 1934. In this communication to Physical Review a forward reference was made to the present paper, the publication of which has been delayed through the inclusion of more applications than was then intended. A brief discussion of the method under consideration was also given by the writer in an earlier paper, Proc. Roy. Soc. 139, 163, (1933).

the representation of the perturbing field by radiation, and then the calculation of the effect of this radiation.

The characteristic features of quantum mechanics enter only into the second part, and, as a result, this kind of analysis makes it easier to see what is the theoretical basis of the final results, and therefore helps us to judge their validity. Interesting cases in this connection are the emission of radiation in collisions between a high energy electron and a nucleus, and the production of pairs by a high-energy photon in a nuclear field. In these cases we find that the quantum mechanics which enters into the existing treatments really concerns only energies of the order of mc^2 however big the energy of the electron or photon.

In connecting up collision phenomena with radiative effects certain features of the former are readily seen to be due to quite familiar results in radiation theory. For instance the large number of collisions, according to Bethe's calculations using Born's method, in which a fast particle looses energy close to the ionisation potential of the atoms traversed, is seen to be exactly the same effect as the well-known rapid increase in photoelectric absorption as an absorption edge is approached. As another example, the much greater radiation emitted by a very fast electron in a classical collision than in a quantum-mechanical collision, is found to correspond very closely to the much greater scattering given by the classical Thomson formula for high frequencies than is given by the quantum-mechanical formula of Klein and Nishina. Another instance, where the ordinary treatment gives no obvious interpretation, is the size of the region around the nucleus from which pairs are produced by high energy photons. This region extends out to a distance of the order of $\xi(h/mc)$ from the nucleus, where

ξmc^2 is the energy of the photon. The present treatment shows that this is so because out to $\xi(h/mc)$, but no further, the field of the nucleus has FOURIER components with high enough frequency to satisfy the threshold energy condition for the production of pairs.

The paper is arranged into two parts. In the first part the conditions of applicability of the method, and the form of the spectrum of the "equivalent" radiation are considered. The second part is devoted to applications, the following problems being considered:

1) Ionisation of atoms.
2) Disintegration of atomic nuclei by fast electrons.
3) Radiative collisions, low velocities, $v/c \ll 1$.
4) Radiative collisions, high velocities, $1 - v/c \ll 1$.
5) Production of electron-pairs by a high energy photon in the field of an atomic nucleus.
6) Production of electron-pairs in collisions between two electric particles.
7) Splitting of a photon in the field of an electric particle.

PART 1. GENERAL

§ 1. Conditions of Applicability.

The general justification of the method under consideration lies in the fact that quantum-mechanics makes no *a priori* distinction between the effect of the field of a charged particle and that of a radiation field. If within a certain region the one has the same electromagnetic description as the other, then they produce the same effects within that region, provided reaction is small. The conditions of applicability of the method are then:

(I) the condition for reproducing the field of the perturbing particle over the region occupied by the perturbed system, during the collision, by a radiation field.

(II) the conditions for considering as independent the effects of the different Fourier frequencies in this field.

As regards the second condition, it follows from the linearity of the equations of motion that the effect of a given 'frequency' in the Fourier spectrum of the perturbing field depends on the other frequencies only if the latter are sufficiently intense that they produce, in a given collision, an appreciable change in the perturbed system. The condition is therefore satisfied if the field of the perturbing particle is sufficiently weak that in a given collision there is only a small probability that the perturbed system makes a transition from its initial state.[1] If this probability (π) is not small then the change produced in the system by a given frequency, ν, may depend to an appreciable extent on the change which is being produced in the system by the other frequencies. The phase-relationship of the different frequencies — which makes them build up into a pulse — is then of primary importance. It should be emphasised, in passing, that if under certain conditions the linearity of the equations of motion required by quantum-mechanics breaks down the separate harmonic components of the perturbing field may not be independent even though $\pi << 1$. The observed failure of the quantum-mechanical radiative formula for very high energy electrons seems to indicate that such conditions do exist if the perturbing field acting on an initially stationary electron is so contracted by the LORENTZ-FITZGERALD effect that its 'thickness' is less than the classical electron radius, e^2/mc^2. This has been discussed elsewhere by the writer[2], and also by OPPENHEIMER[3], and will not be entered into further in this paper.

It is important to note that if we are not concerned with the final state of the perturbed system, the condition $\pi << 1$ need not necessarily be satisfied. This is so for a radiative collision, where we are interested in the radiation emitted by an electron rather than in its final motion after the collision. In this case it is only necessary that the probability be small that in a given collision the perturbed electron, if initially at rest, does not acquire a velocity

[1] This condition ($\pi << 1$) is essentially equivalent to the condition of applicability of the first approximation in BORN's method of treating collisions. In both cases the probability that the perturbed system is left in its initial state, in a given collision, must be large.

[2] E. J. WILLIAMS, Phys. Rev. 45, 729, (1934).

[3] J. R. OPPENHEIMER, Phys. Rev. 45, 903 (1934).

comparable with that of light, in order that its scattering power may remain unaffected.

As regards the reproduction of the field of the perturbing particle by radiation — condition (I) — let us first consider the field at a point P due to a particle A, of charge E, moving with uniform velocity v along aa'. The field at P may be resolved into two components of electric force, $E_1(t)$ and $E_2(t)$, perpendicular and parallel to v respectively; and a magnetic force $H(t)$ equal to $(v/c)|E_1|$, and perpendicular to v and E_1. $E_1(t)$ and $E_2(t)$ are given by

$$E_1(t) = \xi E p \{p^2 + \xi^2 v^2 t^2\}^{-3/2}$$

$$E_2(t) = \xi E v t \{p^2 + \xi^2 v^2 t^2\}^{-3/2}, \quad \xi = (1 - v^2/c^2)^{-1/2} \quad (1)$$

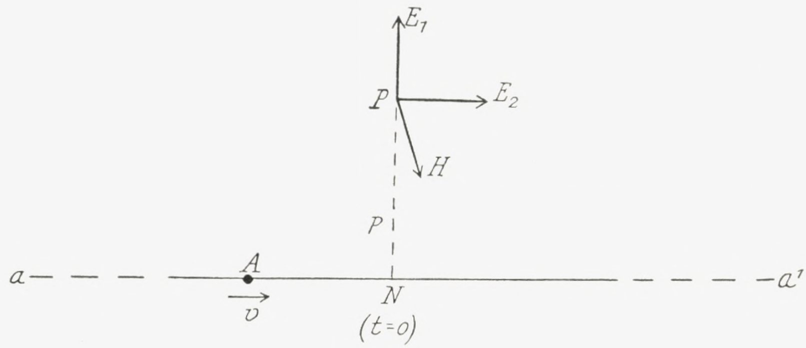

Fig. 1.

It will be noticed that the interval of time, T, for which the forces are appreciable is of the order of $p/\xi v$, and that during this time $|E_2| = |E_1| vt/p \approx |E_1| (1 - v^2/c^2)^{1/2}$.

Let us now imagine a plane radiation pulse R_1 travelling parallel to v, with an electric force equal to $E_1(t)$ at P; and another pulse R_2 travelling perpendicular to v and E_1, with an electric force equal to $E_2(t)$. These radiation pulses accurately reproduce the electric field of A at P, but not its magnetic field. The magnetic field of the pulses exceeds that of A by a component equal to $|E_1|(1-v/c)$ parallel to H, and a component equal to $|E_2|, \approx |E_1|(1-v^2/c^2)^{1/2}$, parallel to E_1. Since the relative importance of the magnetic and electric forces in a perturbing electromagnetic field is of the order of u^2/c^2, where u is the velocity of the perturbed particle, it follows that the errors arising from the misrepresentation of the magnetic field of A by the pulses R_1 and R_2, is of the order of $(1-v^2/c^2)\,u^2/c^2$. They may therefore be neglected provided

$$(1-v^2/c^2)\,u^2/c^2 << 1. \qquad (2)$$

This condition is satisfied in most collision problems because velocities u comparable with c do not usually occur in the perturbed system except when the velocity v of the perturbing particle, A, is close to c.

We have chosen the pulses R_1 and R_2 to represent the field of the particle A only at one point P, and for only one velocity of the observer, viz., zero velocity in the system in which A moves with velocity v. In order that they may have the same effect as A in actual collisions, they must reproduce the field of A for an observer situated at any of the points where the perturbed particle may be during the collision, and travelling with any of the velocities which it may have. It may be shown that these conditions are satisfied provided

(a) $L << p$, where L represents the dimensions of the perturbed system at the beginning of a collision, p the impact parameter.

(b) $u(1-u^2/c^2)^{-1/2} << v(1-v^2/c^2)^{-1/2}$. u denotes the range of velocities of the perturbed particle, including velocities acquired during the collision under consideration. v denotes the velocity of the perturbing particle. Both u and v are measured in a system in which the perturbed particle is initially at rest (or more precisely in which its average initial velocity is zero).[1]

Finally we must consider the conditions for regarding the perturbing particle, A, as a centre of force moving with uniform velocity. Deviations from this will occour for two reasons, viz. the reaction of the perturbed system on A, and secondly the natural uncertainty in its position and velocity. In nearly all problems to be considered here the perturbed particle is an electron, so that the mass, m, of the perturbed particle is never greater than the mass, M, of the perturbing particle, A. It follows that in a collision which satisfies the above condition (b) the transfer of momentum during the collision is small compared with the momentum of A. The effect of reaction can therefore be neglected if (b) is satisfied. The effect of the natural uncertainty in the velocity and position of A on the definition of its path is small provided its de Broglie wavelength, h/Mv, is much less than the impact parameter p. This is also satisfied if the conditions (a) and (b), given above, are fulfilled, because if $L << p$ and $u << v$ then we must have $h/mv << p$, and therefore $h/Mv << p$, since $M >> m$. (a) and (b), and the condition for the independence of the Fourier components of the perturbing field of A, thus represent the complete conditions for the applicability of the method.

§ 2. Spectrum of Equivalent Radiation.

The spectral distribution in the pulses R_1 and R_2, which represent the field of the perturbing particle in a collision with an impact parameter, p, is given in terms of the ex-

[1] This condition ensures that the condition (2), regarding the representation of the magnetic field, is also satisfied.

pressions (1) for the component forces, $E_1(t)$, and $E_2(t)$, by the formula

$$I_{1,2}(\nu) = (c/2\pi)\left|\int E_{1,2}(t)\, e^{2\pi i\nu t}\, dt\right|^2 \tag{3}$$

$I(\nu)$ is in ergs, per unit frequency range, per unit area of the plane of the pulse. For low frequencies such that $\nu << v\xi/p$, the exponential in (3) varies very little during

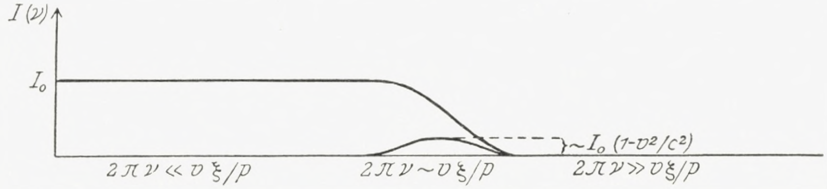

Fig. 2.

the time, T, for which E_1 and E_2 (given by (1)) are appreciable. For such frequencies, $I(\nu)$ is therefore independent of ν, and is given by

$$I_{1,2}(\nu) = I_0 = (c/2\pi)\left\{\int E_{1,2}\, dt\right\}^2. \tag{4}$$

Using the expressions (1) this gives $I_0 = (2c/\pi)E^2/p^2v^2$ for R_1, and zero for R_2 (E_2 being an odd function of the time).

For high frequencies such that $2\pi\nu >> v\xi/p$, E_1 and E_2 vary very little during one period of the exponential in (2), so that $I(\nu)$ is negligible for both R_1 and R_2. Thus for R_1, $I(\nu)$ is constant for $2\pi\nu << v\xi/p$, falls off rapidly in the region $2\pi\nu \sim v\xi/p$, and is negligible for $2\pi\nu >> v\xi/p$. R_2 is appreciable only for $2\pi\nu \approx v\xi/p$, and in this region its intensity relative to that of R_1 is of the order of $(1 - v^2/c^2)$. These main features are represented schematically in Fig. 2.

In actual applications of the method it is convenient to have an expression for the spectrum of the resultant radiation which represents the effect of the perturbing particle in all collisions with impact parameter greater than some quantity p_m. This is given in terms of the intensity $I(\nu, p)$ of the pulses R_1 and R_2, for a given value of the impact parameter, p, by the integral

$$I(\nu, p_m) = \int_{p_m}^{\infty} I(\nu, p)\, 2\pi p dp$$

Using (1) for $E_1(t)$ and $E_2(t)$, and (2) for $I(\nu, p)$ this gives[1]

$$\left.\begin{aligned} I_1(\nu, p_m) &= (4E^2 c/v^2) \log\left(1\cdot 12\, v\xi/2\pi\nu p_m \sqrt{\varepsilon}\right) \\ I_2(\nu, p_m) &= (2E^2 c/v^2)(1 - v^2/c^2) \\ I(\nu, p_m) &= I_1 + I_2 = (4E^2 c/v^2) \log(fv\xi/2\pi\nu p_m), \\ f &= 1\cdot 12\, \varepsilon^{-v^2/2c^2} \approx 1. \end{aligned}\right\} \quad (6)$$

The expression for the resultant intensity falls off logarithmically with the frequency ν, and $(v\xi/p_m)$ is an effective upper limit to the spectrum. The formula is not accurate near the upper limit, but in most of the present applications such frequencies are not involved.

The units in term of which (6) is expressed are such that $I(\nu, p_m)\, d\nu$ is the energy of the equivalent radiation of frequency between ν and $\nu + d\nu$ per particle. It corresponds to a number of photons in the interval $d\nu$ equal to

$$\begin{aligned} N(\nu, p_m)\, d\nu &= (4E^2 c/v^2 h\nu) \log(fv\xi/2\pi\nu p_m)\, d\nu \\ &= (2/\pi)\, \alpha z^2 (v/c)^{-2} \log(fv\xi/2\pi\nu p_m)\, d\nu/\nu \end{aligned}$$

where

$$\alpha = e^2/\hbar c, (= 1/137), ze = E. \quad (7)$$

This formula represents the effect of the field of the moving particle at distances from its path greater than p_m. The suitability of the method in any given problem depends

[1] Cf. N. Bohr, Phil. mag. 25, 10, 1913; 30, 531, (1915).

on whether or not a value of p_m can be chosen so that the effect of the field of the particle inside p_m is of little importance, while at the same time its field outside p_m can be represented by radiation, in accordance with the conditions (a) and (b) of § 2. This is approximately so in the problems considered in this paper. We shall here consider in particular the case of those problems, such as the radiative effect and "pair-production", in which the perturbed particle is an electron and the perturbing particle an atomic nucleus, whose charge ze, we shall assume to satisfy $ze^2/hv << 1$. In such problems the nuclear field inside a sphere around the nucleus of radius less than $r_0 \sim \hbar/mv$ makes no significant addition to the resultant effect. As v approaches c the radius r_0 of this sphere, inside which the field is of little importance, approaches $\hbar/mc$, and not the DE BROGLIE wavelength $(h/mc)(1-v^2/c^2)^{1/2}$, though it is the latter which defines the ineffective region for scattering. The reason for this will not be entered into here as it is considered in a later paper, on some general collision problems, by Professor BOHR and the writer. Here we need only consider to what extent the nuclear field outside $\hbar/mv$ can be represented by radiation.

For the representation of the field at a given distance p from the nucleus, by radiation, conditions (a) and (b) require the perturbed electron in a collision with impact parameter p to be, during the collision, within a volume of dimensions $L << p$, and not to have momentum approaching $mv(1-v^2/c^2)^{-1/2}$ — both L and the momentum being measured in a system in which the average initial velocity of the electron is zero[1]. Now if we represent the

[1] These conditions are of course also conditions for the use of the idea of impact parameter.

electron at the beginning of such a collision by a wave-packet of dimensions, $L, \sim (\hbar/mv\cdot p)^{1/2}$ then the ranges of position and velocity during the collision, due to the natural uncertainty associated with this finite wave-packet, are of the order of $(\hbar/mv\cdot p)^{1/2}$ and $(\hbar/mv/p)^{1/2}\, v$ respectively. Therefore, apart from velocities acquired during the collision, the conditions (a) and (b) are sensibly satisfied provided $p > \hbar/mv$. As regards the velocities acquired during the collision these may be due to the ordinary process of momentum transfer, or to the special phenomenon under consideration. In the radiative effect for example, the electron acquires recoil momentum through the emission of radiation. This second kind of momentum transfer will be considered in the actual applications. The first kind of momentum transfer effectively takes place in every collision and its average value is equal to the ordinary classical momentum transfer. The latter corresponds to a change of velocity of the order of ze^2/pvm. For $p \sim \hbar/mv$ this is equal to $(ze^2/\hbar v)\cdot v$, which is small compared with v under the condition $ze^2/hv << 1$. Provided therefore that $(ze^2/hv) << 1$, the field of the nucleus outside $\hbar/mv$ can, to a first approximation, be represented by radiation. The spectral distribution of energy in the equivalent radiation is then given by (7) with $p_m = \hbar/mv$. With this value of p_m (7) becomes

$$N(\nu)\,d\nu = (2/\pi)\,\alpha z^2\,(c/v)^2 \log\,(gmv^2\xi/h\nu)\,d\nu/\nu. \qquad (8)$$

This formula will be used in many of the subsequent applications. We cannot give a definite value to the numerical coefficient g inside the logarithmic term. We only know that it is of the order of unity. The reason for this is that the lower limit to the effective region of the nuclear field in the problems concerned can only be said to be of the order of $\hbar/mv$, and also the fulfilment of the conditions (a) and (b) is critical in the limit $p \sim \hbar/mv$. This uncertainty regarding the value of g represents the degree of approximation attained by the method in these problems. When the order of magnitude of the argument of the logarithmic term is large, as is the case in many of the problems considered, the approximation is good.

It might be mentioned here that the fine-structure constant, α, which enters into the equations (7) and (8), and the logarithmic term, are the principal factors in the relation of the effects of an electric particle to those of radiation. The logarithmic term depends essentially on the Coulomb law of force. $(\xi v/2\pi\nu)$ in the argument of this term in (7), represents the greatest distance from the path of a particle moving with velocity v at which frequencies ν are found in the spectrum of its field (see fig. 2). If the perturbing field is not a Coulomb field then the logarithmic term must be modified. A case of practical interest, and where a modification can readily be made, is when the perturbing particle is an atomic nucleus, shielded by the atomic electrons. In that case, if the quantity $(\xi v/2\pi\nu)$ inside the logarithmic term is greater than atomic dimensions, it must be replaced by a "shielding" radius. For atomic number Z the effective shielding radius is of the order of $Z^{-1/3}a = Z^{-1/3}(\hbar/mc\alpha)$, a being the hydrogen radius. Making the modification in the case of formula (8) we have

$$N(\nu)\,d\nu = (2/\pi)\,\alpha z^2\,(c/v)^2 \log\,(v/\alpha c z^{1/3})\,. \tag{9}$$

As regards the polarisation of the "equivalent" radiation represented by the above equations the photons contributed by the pulse R_1, corresponding to the component of electric force in the field of the particle perpendicular to its path, are effectively unpolarised and travel in the same direction as the electric particle. They are unpolarised because the resultant photon distribution involves an integration round the path of the moving particle as axis. When $(1 - v^2/c^2) << 1$ the pulse R_2 is of negligible intensity, so that under such conditions the whole of the equivalent radiation is represented by R_1, and is therefore unpolarised and travels with the particle. R_2, though always small, is not negligible, if $(1 - v/c) \sim 1$. Under such conditions the resultant equivalent radiation is partially polarised because the electric force in R_2 is always parallel to the path of the particle. This must be remembered if a polarisation is of importance.

PART 2. APPLICATIONS

§ 3. General relations.

We shall here consider the general relations between various collision phenomena and radiation effects. These relations form the basis of the more detailed calculations given in the sections to follow. The simplest example is the ionisation (or excitation) of an atom by a moving particle. This effect is related to the photoelectric action of radiation, and may be calculated by considering the photoelectric absorption of the "equivalent" radiation discussed above. The process may be represented as follows:

$$S_{occ} + S'_{emp} + h\nu_p \rightarrow S_{emp} + S'_{occ}. \tag{10}$$

S_{occ} denotes an occupied state in the atom, S'_{emp} an empty state of higher energy. $h\nu_p$ denotes a photon which we associate with a frequency ν_p in the FOURIER spectrum of the field of the moving particle, — we can call it a virtual photon.

The reverse of an ionising collision is a collision of the second kind, in which the atom is initially in an excited state, and drops to a state of lower energy during the collision — the colliding particle experiencing a gain of energy. In terms of radiation effects this process is one of induced emission, and may be represented by

$$S'_{occ} + S_{emp} + Nh\nu_p \rightarrow S'_{emp} + S_{occ} + (N+1)\,h\nu_p. \quad (11)$$

N is a measure of the number of virtual photons, per unit frequency range in the field of the perturbing particle, whose frequencies correspond to the energy difference ε between the states S and S'. Owing to the induced nature of the phenomenon the quantum of energy given out by the atom is taken up by the moving particle, and does not appear as a free radiation photon. This is indicated in (10) by $(N+1)\,h\nu_p$ on the R. H. S.[1]

The relations between other collision phenomena and radiation effects are given in the following table. In all the cases the perturbing particle is an atomic nucleus, and the perturbed particle an electron. The notation used in the third column is analogous to that used in (10) and (11). $\bar{S}$ denotes a Dirac negative energy state, $\overset{+}{S}$ an ordinary state of positive energy. $\bar{S}_{emp}$ accordingly means a positive electron, and, $\overset{+}{S}_{occ}$ a negative electron. A photon of frequency ν_n in the virtual radiation field of the nucleus is denoted by $h\nu_n$, an ordinary (or external) photon by $h\nu_e$.

It might be mentioned that though many of the effects given in the above table concern interaction with electrons in negative energy states it does not seem possible to include with them the coherent scattering of radiation in the

[1] It is of course not meant that the new spectrum of the particle corresponds to such a change in the intensity of a given frequency. Actually the acquisition of energy by the moving particle affects the whole of its Fourier spectrum. This difference between the field of a moving particle and a true radiation field is one of reaction, and is of no consequence provided the energy exchange is small compared with the kinetic energy of the particle (cf. § 1). If the effect of radiation essentially depended upon its quantisation, this would not necessarily be true. That is, however, not the case, the effect of radiation being calculable quantum-mechanically from its electro-magnetic description alone, without explicit reference to photons.

Relations between collision phenomena and radiation effects.

Collision Phenomenon	Corresponding Radiation Effect	Equation for Process
1) Radiative Collision.	Scattering.	$\overset{+}{S}_{occ} + \overset{+}{S}'_{emp} + h\nu_n \rightarrow$ $\overset{+}{S}_{emp} + \overset{+}{S}'_{occ} + h\nu_e$
2) Reverse of Radiative Collision.	Induced Scattering.	$\overset{+}{S}'_{occ} + \overset{+}{S}_{emp} + h\nu_e + Nh\nu_n \rightarrow$ $\overset{+}{S}'_{emp} + \overset{+}{S}_{occ} + (N+1)\, h\nu_n$
3) Pair-Production by photon, $h\nu_e$, in nuclear field.	Pair-production in free space by 2 photons.	$\overset{-}{S}_{occ} + \overset{+}{S}_{emp} + h\nu_e + h\nu_n \rightarrow$ $\overset{-}{S}_{emp} + \overset{+}{S}_{occ}$
4) Pair-Production by two particles.		$\overset{-}{S}_{occ} + \overset{+}{S}_{emp} + h\nu'_n + h\nu''_n \rightarrow$ $\overset{-}{S}_{emp} + \overset{+}{S}_{occ}$
5) Annihilation of pairs in nuclear field giving one photon.	Induced annihilation of a pair by radiation, giving two photons (one of these being identical with the inducing radiation).	$\overset{-}{S}_{emp} + \overset{+}{S}_{occ} + Nh\nu_n \rightarrow$ $\overset{-}{S}_{occ} + \overset{+}{S}_{emp} + (N+1)\, h\nu_n + h\nu_e$
6) Splitting of photon ($h\nu_e$) into two photons ($h\nu'_e$ and $h\nu''_e$) in nuclear field.	Simultaneous scattering of 2 photons by electrons in negative energy states.	$h\nu_e + h\nu_n \rightarrow$ $h\nu'_e + h\nu''_e$
7) Radiative collision due to simultaneous action of 2 particles on electrons in negative energy state.		$h\nu_{n1} + h\nu_{n2} \rightarrow$ $h\nu'_e + h\nu''_e$

field of a nucleus. The essential feature of such scattering is that we have one photon before the interaction, and afterwards one photon travelling in a different direction. The change in direction means a momentum transfer, and if the nuclear field was equivalent to radiation, this could not take place without one of the photons in the "nuclear" radiation being deflected. This would result in 2 photons after the encounter, and would therefore not represent coherent scattering. It therefore appears that coherent scattering essentially depends on the difference between the nuclear field and a radiation field. This difference vanishes as the energy of motion of the perturbed particle relative to the nucleus becomes large compared with mc^2. This would indicate that there is only a small contribution to coherent scattering from the production of virtual pairs of positive and negative electrons with energies large compared with mc^2.

§ 4. Excitation and Ionisation of Atoms.

The ionisation and excitation of atoms by electric particles has been considered extensively by various workers on the basis of Born's theory of collisions. The case of particles whose velocity is large compared with the orbital velocity of the electrons in the atoms traversed has, in particular, been accurately worked out by Bethe[1]. The problem is considered here only as an example of the use of the method described in the first part of this paper, and to show how some of the main features of the problem considered by Bethe readily follow from certain well-known properties of the absorption of radiation. The consideration of this problem also makes clearer the treatment of the analogous problem of the disintegration and excitation of

[1] H. Bethe, Ann. Physik, 5, 325, (1930).

atomic nuclei by electrons, which is considered in the next section.

A complete treatment of the ionisation and excitation produced by an electric particle by the method of replacing its field with radiation is not possible, because the conditions set out in § 1 are not satisfied in close collisions in which the particle passes through the atom (impact parameter, $p <$ atomic dimensions, d). The method can be applied only to more distant collisions in which the path of the particle lies outside the atom ($p > d$). We shall see, however, that the frequency of excitation and of ionisation is mainly due to these collisions. The close collisions are of a different type and are responsible only for a comparatively small number of large energy transfers to the atomic electron.

We shall consider, for definiteness, the ionisation, by electrons, of hydrogen-like atoms in the ground state (nuclear charge ze), and assume that the velocity v of the incident electron is large compared with the orbital velocity, u, of the atomic electron, i. e. $ze^2/\hbar v << 1$.

The intensity of the virtual radiation which represents the field of an electron in collisions with impact parameter greater than atomic dimensions, d, is given by (7) § 1, if we substitute d for p_m in that equation. As p_m occurs inside the logarithmic term in (7) an exact definition of d is not important, and for hydrogen atoms we can take it equal to the diameter of the hydrogen Bohr-orbit, i. e. $(\hbar^2/2mJ)^{1/2}$, J being the ionisation potential. Making this substitution for p_m in (7) we have, for the distribution of photons in the corresponding radiation,

$$N(\nu)\,d\nu = \pi^{-1}\alpha\,(c^2/v^2)\log\left(gmv^2J/Q^2\,(1-v^2/c^2)^{1/2}\right)d\nu/\nu \quad (12)$$

$$g = 0\cdot 56 \sim 1.$$

The ionisation produced by the electron is now obtained by considering the photoelectric absorption of the virtual radiation represented by (12), and the excitation may be obtained by considering the line-absorption.[1] Actually the probability $\Phi_1(Q)\,dQ$ of ionisation, in which the energy transfer lies between Q and $Q+dQ$ is the product of (12), with $\nu = Q/h$, and the atomic absorption coefficient, μ, of the matter traversed for radiation of frequency $\nu = Q/h$. Thus

$$\Phi_1(Q)\,dQ = N(Q/h)\,d(Q/h)\cdot\mu(Q/h). \qquad (13)$$

The theoretical value of μ has been accuralety evaluated for hydrogen atoms. The exact formula is, however, rather complicated and to see more readily the form of $\Phi_1(Q)$ we shall here use the following approximate expression for μ viz.,

$$\mu(\nu) = 0.77\,\pi\,(e^2/mc)\,(J/h)^{1.8}\,\nu^{-2.8}. \qquad (14)$$

Since $J\alpha \sim Z^2$ this corresponds very nearly to the well-known "$Z^4\lambda^3$" law of photoelectric absorption. Substituting in (13) we obtain

$$\begin{aligned}\Phi_1(Q)\,dQ = 0.77\,(2\pi e^4/mv^2J^2)\,(J/Q)^{3.8} \\ \log\left(gmv^2J/Q^2(1-v^2/c^2)\right)dQ.\end{aligned} \qquad (15)$$

To obtain the total effect of the moving electron we must add to (15) the effect of close collisions in which the electron passes through the atom ($p < d$). These collisions can be very simply treated, being practically equivalent to collisions between two free electrons in which their distance of approach is of the order of d and less. The effect of such collisions between two free electrons can be readily

[1] There will also be some ionisation and excitation due to the Compton and Raman scattering of the virtual radiation, but this is quantitatively unimportant. Such scattering represents a type of radiative collision, and is considered separately in § 6.

shown to be represented by the RUTHERFORD scattering law provided we leave out the scattering through angles corresponding to momentum transfers of the order of or less than h/d. This means that the close collisions, with $p > d$, give rise to energy losses, Q, ranging from the maximum energy loss of $\frac{1}{2}mv^2$ down to about $Q \sim J$, and distributed according to the law for collisions with free electrons, viz.,

$$\left.\begin{array}{l}\Phi_2(Q)\,dQ = (2\pi e^4/mv^2) \\ \quad (1+4J/3Q)\,dQ/Q^2.\end{array}\right\}(16)^1$$

For $Q \sim J$ the formula is of course correct only in order of magnitude. However, in this region of Q, we find that $\Phi_2(Q)$ is small compared with $\Phi_1(Q)$. This brings us to the result that practically all the excitation, and most of the ionising collisions with energy losses of the order of J, are due to the photoelectric effect represented by equation (15).

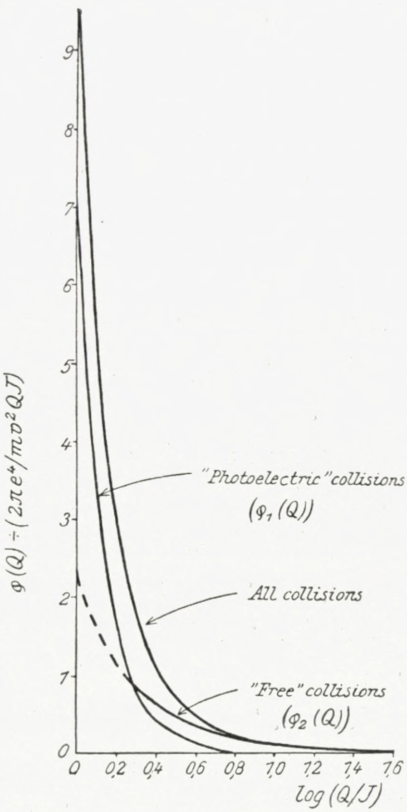

Fig. 3.

The distributions $\Phi_1(Q)$ and $\Phi_2(Q)$ are represented in fig. 3 for electrons of energy $\frac{1}{2}mv^2 = 2{,}500\,J$.

The ordinates represent $\Phi(Q) \div (2\pi e^4/mv^2 JQ)$, and the abscissae log (Q/J), so that the area under the curves is proportional to the number of collisions. The sum of $\Phi_1(Q)$

[1] The second factor in brackets represents the effect of the motion of the atomic electrons.

and $\Phi_2(Q)$, agrees satisfactorily with the distribution due to all collisions as calculated by Bethe using Born's theory. It will be noticed that while both $\Phi_1(Q)$ and $\Phi_2(Q)$ increase markedly with decreasing Q, the very rapid rise in the resultant curve close to J is mainly due to the variation of $\Phi_1(Q)$. We thus see that the large concentration of energy losses close to the ionisation potential which is required by Bethe's calculations is due to the photoelectric action of the field of the particle in distant collisions. The derivation of (15) shows that this concentration is mainly the result of the well-known rapid increase in photoelectric absorption as an absorption edge is approached. It is, however, also partly due to the fact that the density of virtual photons corresponding to the field of the particle varies as ν^{-1}.

It may be of interest to mention that according to recent calculations by Wheeler[1] and others, the photoelectric absorption does not fall off as rapidly with increasing frequency for helium as for hydrogen. The energy losses suffered by a fast particle in helium are therefore not so concentrated near the ionisation potential as in the case of hydrogen. These characteristics of helium somewhat reduce its stopping power, and an outstanding discrepancy of about 10 % between the observed stopping-power of helium and the value calculated assuming hydrogen-like wave-functions[2] may be accounted for in this way.

The division of the inelastic collisions of an electron with an atom, into the two classes represented by $\Phi_1(Q)$ and $\Phi_2(Q)$, is of course not perfectly sharp, the two kinds of collisions actually merging into each other. There is,

[1] J. A. Wheeler, Phys. Rev. 43, 258 (1933).

[2] See e. g. Proc. Roy. Soc. 135, 108 (1932).

however, in general a fairly clear distinction between them, and for a given energy loss this distinction is well exemplified by the momentum relations obeyed. The photoelectric collisions represented by $\Phi_1(Q)$ are essentially 3-body collisions involving the nucleus. By comparison with the photoelectric effect of ordinary radiation, (in which the momentum of the photoelectron is mainly balanced by a "recoil" of the nucleus) we would expect the deflection of the electron in these collisions to be much less than if it were reacted upon by the full momentum of the ejected electron. On

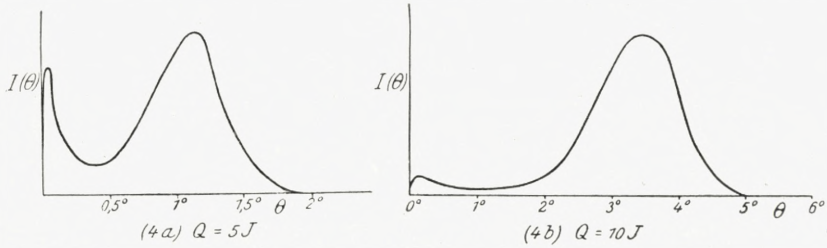

Fig. 4.

the other hand in the two-body collisions, represented by $\Phi_2(Q)$, the ionising electron takes the full reaction. These features are clearly present in the results obtained by BETHE[1] using the general method of BORN. For we find from BETHE's calculations that the angular distribution of electrons which have suffered a *given* energy loss, Q, has two peaks, whose relative sizes are approximately in the ratio of $\Phi_1(Q)$ to $\Phi_2(Q)$[2]. The peak at large angles corresponds to $\Phi_2(Q)$, and its maximum obeys the relation betveen deflection and energy loss for two-body collisions. The other peak, whose magnitude is represented by $\Phi_1(Q)$, is in the region of much

[1] H. BETHE, Ann. Physik, 5, 325, (1930).

[2] The existence of these two peaks has not been previously considered. It seems to be assumed in previous discussions that the peak at large angles falls uniformly with decreasing angle and that there is no second maximum at small angles.

smaller angles. Fig. 4a represents the distribution, according to Bethe's calculations, for $Q = 5J$, and $\frac{1}{2}mv^2 = 2{,}500J$. 4b represents the case of $Q = 10J$. The much smaller relative size of the peak at small angles for $\Phi = 10J$ corresponds to the much more rapid decrease of $\Phi_1(Q)$ with Q than of $\Phi_2(Q)$ with Q.

The angular distribution of the ejected atomic electrons in the photoelectric collisions represented by equation (15) is of particular interest in connection with the one-sidedness of the field acting on the atoms in these collisions. In a distant collision (impact parameter $p >$ atomic dimensions) the component, E_1, of the electric force acting on the atom normal to the path of the electron, is directed away from the path throughout the collision, fig. 1. The electric force in any individual Fourier component in the Fourier representation of E_1 is, however, quite symmetrical as regards the normal to the path of the electron. The "photoelectrons" produced by the action of any one such component are therefore also symmetrically distributed, being directed towards the path of the electron as much as away from it. In analysing the force, E_1, into Fourier components, and assuming these to act independently, it would therefore appear that an essential feature of the perturbation is lost. This is, however, not the case, and the symmetrical distribution of the photoelectrons is quite in accordance with the results obtained if no resolution into Fourier components is made. For if a uniform electric force E_1 acts for a certain time on an atom in a fixed direction x, the probability of the excitation of any given state $\psi_2(r)$ is proportional to

$$\left| \int\int E_1(t)\, x e^{2\pi(w_2 - w_1)t/h}\, \psi_2(r)\, \psi(r,t)\, dt dT \right|^2. \qquad (17)$$

$\psi(rt)$ represents the state of the atomic electron at time t, and depends on the perturbing fore $E_1(t)$. If, however, the perturbation is small we can take $\psi(r,t)$ as constant, and equal to the ψ-function representing the undisturbed atom. In that case the force $E_1(t)$ occurs only in one place in (17), and since this is a square term its value is unaltered if we change the sign of $E_1(t)$. This means that the excited states, which may of course be states in the continuous spectrum and therefore representing photoelectrons, are symmetrical with respect to the positive and negative directions of the electric fore E_1, though E_1 may be directed along only one of these directions throughout the perturbation.

The above result assumes that the perturbation is sufficiently small that the atom may be taken to be in its initial state throughout the collision. This is, however, also the condition for considering the FOURIER components independently (§ 1). The condition is satisfied if $Q_{cl} << \varepsilon$, where Q_{cl} represents the classical energy acquired by a free electron in the same field, and ε represents the energy difference between the initial state of the atom and the next excited state. It is interesting that for a free electron ε is infinitesimal, so that the condition is never satisfied. For a free electron the energy transfer is effectively due to the action of an infinite number of components of zero frequency. The phase relations between these components makes them build up to a one-sided pulse, and the velocity acquired by the electron has the same one-sidedness as in the classical theory — in accordance with EHRENFEST's theorem.

It might be pointed out that for a bound electron traversed by a one-sided pulse, there is, classically, as well as on the quantum theory, no indication of the one-sidedness of the the pulse in the resultant motion of the electron after the collision. This is so because the motion of the electron, being oscillatory, cannot on the average be directed in the direction of the perturbing force any more than in the opposite direction.[1]

§ 5. Excitation and Disintegration of Atomic nuclei by Electrons.

The disintegration of an atomic nucleus by a moving particle *A* may take place through the capture of *A* (with or without the ejection of another particle from the nucleus), or by a process in which *A* only loses some of its energy, being still free after the collision. The second type of disintegration is quite analogous to the ionisation of atoms by electric particles, and can under certain conditions be divided into two classes of collisions on exactly the same basis as the ionisations of atoms considered in the previous section. This means that we can have collisions in which the nucleus is disintegrated by a photoelectric action of the particle *A*, and collisions in which the particle ejected from the nucleus receives the necessary energy by virtually a two-body collision with *A*. The essential condition for this classification is that the velocity, v, of *A* is large compared with the orbital velocity u, in the nucleus, of the particle, *B*, whose ejection from the nucleus we are considering. If the incident particle *A* is an α-particle, or a proton, then in practice this condition is either not satisfied or is very critical. For electrons with sufficient energy to disintegrate

[1] If the motion of the electron during the collision was watched then of course the one-sided nature of the pulse would be observed. The calculation of the effect of the pulse through a Fourier-Integral analysis is, however, only meant to give the resultant state after the collision is over and is not concerned with the actual progress of the collision.

a nucleus the condition is, however, well satisfied, because such electrons have of necessity much greater velocity than the nuclear particles, owing to their much smaller mass. We can thus estimate the probability of nuclear disintegration by electrons by considering separately the effect of 2-body interaction with the nuclear particles, and the effect of "photoelectric" interaction with the nucleus as a whole. The former is actually of little importance, and in fact produces no disintegration at all unless the incident electrons have energy of at least 10^7 volts. This is so because of the small mass of electrons in comparision with the mass of nuclear particles. We need therefore only consider the disintegration produced by the photoelectric type of interaction.

The magnitude of the latter depends in the first place on the intensity of the virtual radiation which represents the relevant part of the field of the electron. This is given by the general expression (7) in § 1, if for p_m we substitute $h/mc\xi$, where $\xi = (1 - v^2/c^2)^{1/2}$, and v is the velocity of the electron.[1] It is assumed here that $h/mc\xi$ is greater than nuclear dimensions, i. e. that the energy of the electron is less than about 100 million volts. If this is not so then p_m must be taken equal to nuclear dimensions, in the same way as it is taken equal to atomic dimensions in the analogous case of the ionisation of an atom.

[1] It can be shown that the field of the electron outside a radius $h/mc\xi$, as regards its effect on the nucleus, can be replaced by radiation in accordance with the conditions set out in § 1, provided $h/mc\xi$ is greater than nuclear dimensions. The conditions are in fact somewhat strained near the lower limit, but, as in other problems, the consequent error in the final formulae concerns only a numerical coefficient in a log. term. As regards the effect of the field inside $h/mc\xi$ we may conclude that it is of no importance, because in a collision with a free particle the energy transfer from the electron is negligibly contributed to by the electronic field inside a radius r if $r << h/mc\xi$.

Making the substitution $h/mc\xi$ for p_m, (7) becomes

$$N(\nu)\,d\nu = (2/\pi)\,\alpha \log(g\,\xi^2\,mc^2/h\,\nu)\,d\nu/\nu. \tag{18}$$

If now we denote by $\sigma(\nu)$ the cross section of the nucleus for disintegration by radiation of frequency ν, and denote the threshold frequency for disintegration by ν_0, the cross section for nuclear disintegration in which the nucleus acquires energy between Q and $Q + dQ$ is

$$\Phi(Q)\,dQ = (2/\pi)\,\alpha \log(g\,\xi^2\,mc^2/Q)\,\sigma(Q/h)\,dQ/Q \tag{19}$$

and the total cross section for disintegration by the electron is

$$\Phi(\xi) = (2/\pi)\,\alpha \int_{\nu_0} \log(g\,\xi^2\,mc^2/h\,\nu)\,\sigma(\nu)\,d\nu/\nu. \tag{20}$$

To proceed further we must know the value of the nuclear absorption coefficient $\sigma(\nu)$. Its value for the nucleus of heavy hydrogen has been calculated by Bethe and Peierls[1], but it can be shown by simple arguments that its order of magnitude must in all cases obey the relation

$$\int_{\nu_0}^{\infty} \sigma(\nu)\,d\nu = \pi E^2/Mc \tag{21}$$

where E and M are the charge and mass of the particle whose ejection from the nucleus we are considering.[2] We would also expect from general considerations that $\sigma(\nu)$

[1] H. Bethe and R. Peierls, Proc. Roy. Soc. (in publication).

[2] Actually if we assume that the disintegration of the nucleus consists in the separation of 2 parts whose charge and mass are E, M, and E^1, M^1, respectively, and if there are no states of excitation, then quite accurately

$$\int \sigma(\nu)\,d\nu = (\pi/c)\,(E/M - E^1/M^1)^2\,MM^1/(M + M^1).$$

This result may be derived by simple correspondence arguments. It is only in the special case of $E/M = E^1/M^1$ that this differs greatly from E^2/Mc.

is appreciable only in the neighbourhood of the threshold. This means that the integral in (20) converges so that the integrated cross section is of the order

$$\sigma(\xi) = g_1 \alpha \log(g_2 \xi^2 mc^2/h\nu_0) \cdot (E^2/Mc\nu_0)$$
$$= g_1 (2\pi e^2 E^2/Mc^2 h\nu_0) \log(g_2 \xi^2 mc^2/h\nu_0). \quad (22)$$

g_1 and g_2 are factors of the order of unity. The accuracy of the formula depends on the energy of the electron being appreciably larger than the threshold energy $h\nu_0$. The energy of the electron must, however, not be too large because we have assumed that $h/mc\xi$ is larger than nuclear dimensions. If this is not satisfied then instead of (22) we find by similar calculations that

$$\sigma(\xi) = g_1 (\pi e^2 E^2/Mc^2 h\nu_0) \log(g_3 \xi Mc^2/h\nu_0). \quad (22a)$$

§ 6. Radiative Collisions. — Non Relativistic ($v/c << 1$).

The emission of radiation in a collision between two electric particles is obtained by the method used here by considering the scattering of the radiation with which we may replace the perturbing fields. If the two particles are an electron and an atomic nucleus then the radiation emitted is practically all due to the scattering of the virtual radiation field of the nucleus by the electron. The intensity-distribution of the radiation with which we can replace the nuclear field in this problem has already been deduced in § 1, and is given by (8) viz.

$$N(\nu)\, d\nu = (2/\pi)\, z\alpha^2 (c/v)^2 \log(gmv^2/h\nu).$$

ξ being taken as unity since we assume $v/c << 1$. v is the relative velocity of the electron and the nucleus, and the formula actually refers to the nuclear field as it appears to an observer relative to whom the electron is initially at rest. The cross section $S(\nu)\,d\nu$ for the emission of a photon of frequency between ν and $\nu + d\nu$ is now the product of $N(\nu)\,d\nu$ and the cross section $s(\nu)$ for the scattering of a photon $h\nu$ by an electron. Under the condition $v/c << 1$ the frequencies involved are much less than mc^2/h, so that we can use Thomson's formula for $s(\nu)$ viz. $(8\pi/3)(e^2/mc^2)^2$. This gives

$$\left.\begin{aligned} S(\nu)\,d\nu = (2/\pi)\,\alpha z^2 (c^2/v^2) \log (gmv^2/h\nu)\, d\nu/\nu \\ \cdot (8\pi/3)(e^2/mc^2)^2. \end{aligned}\right\} \quad (26a)$$

This result may be expressed in terms of the energy, $E(\nu)$, of the scattered radiation per unit frequency range per unit length of path of the electron in an atmosphere containing one nucleus per unit volume, by multiplying by $h\nu$, giving finally

$$E(\nu) = (32\pi/3)(z^2 e^6/m^2 v^2 c^3) \log (gmv^2/h\nu). \quad (26)$$

Since the exact value of the factor g in the logarithmic term is not given the formula is accurate only for $mv_2/h\nu >> 1$.[1]

The direct calculation of the radiation emitted in electron-nucleus collisions, using the method of transitions between stationary states, has been carried out by Sommerfeld[2], Sauter[3], and others. With the limitation $h\nu << mv^2$, (26), is in exact agreement with their results.

The derivation of (26) shows that, in the radiative collisions concerned, the form of the spectrum of the radiation

[1] For $h\nu \sim mv^2$ the change in velocity of the electron relative to the nucleus is comparable with the initial relative velocity, v, which, in view of the conditions given in § 1, is another reason why (26) can give only the order of magnitude of $E(\nu)$ for $h\nu \sim mv^2$.

[2] A. Sommerfeld, Ann. Phys. 11, 257 (1931).

[3] F. Sauter, Ann. Phys. 18, 486 (1933).

emitted is just that of the virtual radiation with which we replace the nuclear field, the absolute intensity being simply this multiplied by the scattering coefficient. The dependence of $E(\nu)$ on ν is only through the logarithmic term in the expression for the virtual radiation. This term may be written $\log(p_{max}/p_{min})$, where $p_{max} = (v/2\pi\nu)$, is the maximum distance from the path of the nucleus at which, to an observer at rest relative to the electron, frequencies ν are found in the nuclear field (cf. fig. 2); $p_{min} = \hbar/mv$ is the effective minimum from the nucleus at which its field is important.

It is interesting to consider, on the basis of the present method, the relation of the quantum-mechanical formula (26) for $E(\nu)$ and the classical formula. The latter can be expressed as the product of the virtual radiation of the nucleus and the scattering coefficient, in just the same way as the quantum-mechanical formula. The scattering coefficient $s(\nu)$ is the same in classical theory as in quantum-mechanics, and the difference between the two cases arises from the different values of p_{min} in the logarithmic term in the representation of the nuclear field. The classical value of p_{min} is ze^2/mv^2, as compared with $\hbar/mv$ in the quantum theory. The classical formula is accordingly

$$E(\nu)_{class} = (32\pi/3)(z^2e^6/m^2v^2c^3)\log(g'mv^3/2\pi\nu ze^2) \quad (27),$$

and has an effective upper limit at $h\nu \sim mv^2 \div (ze^2/hv)$.

The quantum-mechanical formula (26) is based on the assumption that $(ze^2/hv) << 1$ i. e. $h/mv >> ze^2/mv^2$. If this condition is reversed i. e. $ze^2/hv >> 1$, then the derivation of (26) breaks down. As discussed in the later paper referred to in the introduction quantum-mechanics justifies, under these conditions, a classical treatment, so that (27) approximately represents both the classical and quantum mechanical requirements.

§ 7. Radiative Collisions — Relativistic $(1-v/c) \ll 1$.

We shall consider three effects, the radiation emitted by a fast electron in collisions with an atomic nucleus, the interference between the perturbations of a fast electron by the different nuclei in a solid, and thirdly the radiation emitted by the atomic electrons due to the passage of a fast electron. The kinetic energy of the electron in these problems may be taken as $(1-v^2/c^2)^{-1/2} mc^2$, $= \xi mc^2$, since $\xi \gg 1$.

(a) Electron-nucleus collisions. The application of the Fourier-Analysis method to this case was considered by the writer in a recent communication to the Physical Review[1], and it has been elsewhere carried out in detail by Weiszäcker[2]. We shall here only make an approximate calculation sufficient for a discussion of the points upon which the radiative formula depends, and to make clear the analogy with the production of positive electrons by high energy photons considered in § 5[3].

As in the non-relativistic case we have to calculate the scattering, by the electron, of the virtual radiation repre-

[1] E. J. Williams, Phys. Rev. 45, 729, (1934).

[2] C. F. v. Weiszäcker, Zeit. f. Phys. 88, 612, (1934).

[3] The present procedure is somehwat different from that used by Weiszäcker, and is the same as that originally used by the writer in an application to radiative collisions (1933). The calculations made then were not sufficiently accurate to reproduce the formula obtained by Heitler and Sauter (Nat. Dec. 9, 1933) but that this could be done was shown by the work of Weiszäcker.

senting the field of the nucleus. We shall consider this scattering in a system S' in which the electron is initially at rest, because the scattering formulae in their usual form refer to stationary electrons. The velocity of the nucleus in this system is v, and since $(1 - v^2/c^2)^{-1/2} >> 1$ the expression (8) for the radiation with which we can replace the nuclear field reduces to

$$N(\nu)\, d\nu = (2/\pi)\, \alpha z^2 \log(\xi mc^2/h\nu)\, d\nu/\nu. \qquad (28)$$

We shall consider the scattering of the virtual photons represented by (28) in two parts, one with frequencies $\nu_n < mc^2/h$, the other with frequencies $\nu_n > mc^2/h$. The suffix n is added to denote the frequency of the photons before scattering. ν' shall denote the frequency after scattering, and ν the scattered frequency in the system in which the nucleus is initially at rest.

For $\nu_n < mc^2/h$ we can, to a first approximation, use THOMSON's formula for the scattering, and also neglect the COMPTON change of wavelength, so that $\nu' = \nu_n$. The number of photons emitted in S' with frequencies between ν' and $\nu' + d\nu'$ is then

$$\left. \begin{aligned} n'(\nu')\, d\nu' = {} & (8\pi/3)(e^2/mc^2)^2 \cdot (2/\pi)\, \alpha z^2 \\ & \log(\xi mc^2/h\nu')\, d\nu'/\nu'. \end{aligned} \right\} \qquad (29)$$

The relation between ν' in S' and the frequency ν in S is

$$\nu = \nu'(1 - v^2/c^2)^{-1/2}(1 - v/c \cos\theta') = \nu' \xi (1 - v/c \cos\theta') \qquad (30)$$

where θ' is the angle between the velocity v and the direction of scattering in S'. Since the THOMSON scattering is symmetrically distributed about $\theta' = \pi/2$, and since, for a given θ', $d\nu/\nu = d\nu'/\nu'$ it follows that the number of scattered photons in the system S, with frequencies between

ν and $\nu + d\nu$, is given by (29) provided we substitute ν for ν', i. e.

$$n(\nu)\, d\nu = (16/3)\,(e^2/mc^2)^2\, \alpha\, z^2 \log(g\,\xi^2\, mc^2/h\nu)\, d\nu/\nu. \quad (31)$$

This result assumes THOMSON scattering and it applies accurately only for $h\nu_n << mc^2$ i. e., $h\nu << \xi mc^2$. In this region of frequencies it agrees exactly with the results obtained by HEITLER, BETHE and SAUTER[1], by direct application of DIRAC's electron theory using the method of transition between stationary states. We thus see that radiative energy losses appreciably less than the incident energy, ξmc^2, of the electron — though possibly much greater than mc^2 — are due to the THOMSON scattering of low frequencies, $\nu_n << mc^2/h$, in the system S'.

Multiplying (31) by $h\nu$ and integrating up to $h\nu_n = mc^2$ i. e. $h\nu \sim \xi mc^2$, we obtain for the total energy loss due to $\nu_n < mc^2/h$

$$R_1 = A_1\, \alpha\, z^2\, (e^2/mc^2)^2 \log(g\,\xi)\, \xi\, mc^2, \quad A_1 \approx 1. \quad (32)$$

For the frequencies $\nu_n > mc^2/h$ the scattering takes place mainly in a direction $\theta' \approx (mc^2/h\nu_n)^{1/2}$, and the scattered frequency is approximately $\nu' = \nu_n \div \{1 + (h\nu_n/mc^2)\,\theta'^2\} \approx a\nu_n$ where a is a fraction not much less than unity. From (30) the scattered frequency in the system S is accordingly

$$\nu = a\nu_n\,\xi\,(1 - v/c \cos\theta') \approx \tfrac{1}{2}\, a\, \xi\, (mc^2/h).$$

Since $a \approx 1$, this means that for every quantum of the nuclear radiation of energy $h\nu_n > mc^2$ scattered in S', a quantum $h\nu \approx \xi mc^2$ is emitted in S. The total energy lost by the electron in the system S, due to $\nu_n > mc^2/h$, is therefore

[1] H. BETHE and W. HEITLER, Proc. Roy. Soc. 146, 83, (1934). — W. HEITLER and F. SAUTER, Nature, 132, 892 (1934).

$$R_2 = \int_{mc^2/h}^{\sim \xi mc^2/h} (e^2/mc^2)^2 (mc^2/h\nu_n) \log (h\nu_n/mc^2) \cdot$$
$$\cdot \alpha z^2 \nu_n^{-1} \log (g \xi mc^2/h\nu_n)\, d\nu_n \cdot \xi mc^2 .$$

The successive factors in the integrand represent respectively the KLEIN-NISHINA scattering coefficient, the spectrum of the virtual radiation, and the energy of the emitted photon in S, all only approximately. Denoting $h\nu_n/mc^2$ by x we have

$$\begin{aligned} R_2 &= A_2 (e^2/mc^2)^2 \alpha z^2 \xi mc^2 \int_1^{\sim \xi} x^{-2} \log x \log (g \xi/x)\, dx \\ &= A_2 (e^2/mc^2)^2 \alpha z^2 \xi mc^2 \log (g \xi), \quad A_2 \sim 1. \end{aligned} \tag{33}$$

R_2 is thus of the same order as R_1, so that the frequencies $\nu_n > mc^2/h$ make no large addition to the radiative energy loss. This may be contrasted with classical theory, according to which the frequencies $\nu_n > mc^2/h$ increase the total effect by about $(\log \xi)^2$. This difference comes from the large difference between the KLEIN-NISHINA scattering formula and the THOMSON formula, for frequencies much above mc^2/h. The THOMSON formula is independent of the frequency, while in this region the KLEIN-NISHINA formula decreases with frequency approximately as $(mc^2/h\nu) \log (h\nu/mc^2)$. This makes the integral in (33) converge fairly rapidly. This convergence means that the relative contribution to the final radiative effect from the scattering, in the system S', of frequencies appreciably greater than mc^2/h is very small. The bearing of this on the validity of the radiative formula has been dicussed elsewhere[1].

The total energy lost by the electron, expressed in terms of the cross-section for losing all its energy, is

[1] C. F. v. WEISZÄCKER, Zeit. f. Phys. 88, 612, (1934). — E. J. WILLIAMS, Phys. Rev. 45, 729, (1934).

$$\sigma = (R_1 + R_2) \div \xi mc^2 = A\,(e^2/mc^2)^2 \alpha z^2 \log(g\xi),\ A \sim 1,\ g \sim 1\,. \quad (34)$$

This is the same as the formula derived by Heitler and Sauter[1] provided we take $A = 4$, $g = 2.1$. The calculations of Weiszäcker[2] show that a more detailed application of the present method to this problem gives also the exact value of A.

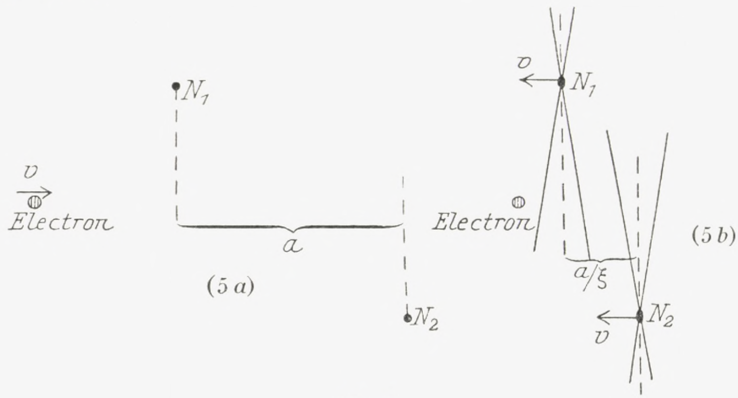

Fig. 5.

Interference between the radiative effects due to different nuclei in a solid. When an electron traverses a solid it encounters the field of different nuclei in succession, and for very high energies of the electron the perturbations due to the different nuclei will not be independent. Let us first consider the effect of two nuclei N_1, N_2, on opposite sides of the path of the electron, and at a distance a apart in a direction parallel to the path of the electron (fig. 5a).

Then in a system S' in which the electron is initially at rest, the electron receives two equal and opposite pulses within a time $a/\xi c$ of each other (fig. 5b). The Fourier spectrum of such a field contains practically no frequencies

[1] W. Heitler and F. Sauter, Nature 132, 892 (1934).
[2] C. F. v. Weizsäcker, Zeit. f. Phys. 88, 612, (1934).

below $c\xi/a$. Since in S' the radiative effect due to a single nucleus comes almost entirely from the scattering of frequencies of the order of mc^2/h and less, it follows that if $c\xi/a > mc^2/h$, i. e. $\xi > amc/h$, the radiation emitted will be negligible compared with that emitted if the two nuclei acted independently. In other words the interference between the two nuclei completely cuts out the frequencies which are important to the radiative effect. The neighbouring nuclei in a solid are about 137 $(\hbar/mc)$ apart, so that the interference effect for frequencies mc^2/h sets in when the energy of the electron (in S) approaches 137 mc^2. The interference effect will, however, not always be a 'cutting out' effect, because there will be cases where successive nuclei are on the same side of the path of the electron, thus enhancing the radiative effect. A closer analysis shows that the electrons must have energy very much greater than 137 mc^2 before the interference between the successive nuclei in a solid sensibly reduces the average radiative energy loss. Assuming the distribution of atomic nuclei in the solid to be quite regular the effect is found to set in at energies of the order of $137 \cdot z^{2/3} mc^2$. This means that for electrons traversing lead, for example, the reduction in the average radiative energy loss due to interference can be neglected unless their energy is of the order of 10^{10} volts.

Radiation from atomic electrons. In a collision between a moving electron and an atom, radiation is emitted by the atomic electrons as well as by the moving electron[1]. It may readily be shown that the former is relatively very small. Its magnitude per cm of path of the

[1] The interference effect beetwen the radiations emitted in a collision between two electrons is unimportant under the condition $1 - v/c << 1$.

moving electron, in an atmosphere containing 1 atom per cc. is

$$R = \int_0^{\xi\, mc^2/h} z N(\nu)\, \sigma(\nu)\, h\nu\, d\nu.$$

$N(\nu)$ is the density of photons in the virtual radiation representing the field of the moving electron $\sigma(\nu)$ is the, Klein-Nishina scattering coefficient per electron, and z is the number of electrons per atom. On integration we find

$$R = (1/3)\, \alpha z\, (e^2/mc^2)^2\, mc^2\, (\log \xi)^3. \qquad (35\,^1)$$

The ratio of this to the radiation emitted by the moving electron, represented by (34), is of the order of $(\log \xi)^3 / z\xi \log(137\, z^{-1/3})$. This ratio is in all cases very small.

§ 8. Production of Electron-Pairs by a High energy Photon in the field of an atomic nucleus.

The annihilation of a pair of positive and negative electrons giving two photons, and the converse process of the creation of an electron-pair by two photons, were the consequences of his theory of the electron first considered by Dirac. As regards the second process it was evident that the intensity of radiation necessary to give a detectable effect would not be available in practice, and it appeared that the theoretical possibility of the creation of electron-pairs was outside experimental test. We now know that this is not so, and that if instead of using two beams of radiation we irradiate ordinary matter with one beam then it is possible to create electron pairs to a measurable degree. Under these conditions a pair is produced not by the action of two photons as considered by Dirac, but by a photon

[1] This also applies to protons of the same velocity, i. e. protons with energy $\xi\, Mc^2$ where M is the mass of the proton.

and an atomic nucleus. If, however, we think of the perturbing field of the nucleus in terms of its FOURIER components, according to the method of the present paper, we find that the two effects are really closely connected. In the case of a photon, whose energy is large compared with mc^2, the connection between the two effects is indeed very close, and the formula for pair-production by such a photon and a nucleus may readily be expressed in terms of the formula for pair-production by two photons.

The latter was recently derived by BREIT and WHEELER[1], and is of course also implied in DIRAC's annihilation formula. For two photons $h\nu_1 = xmc^2$, $h\nu_2 = ymc^2$, travelling in opposite directions, BREIT and WHEELER's result, in terms of cross section is

$$\sigma(x,y) = \pi(e^2/mc^2)^2\{2\theta u^{-2} + 2\theta u^{-4} - \theta u^{-6} - (u^2-1)^{1/2}(u^{-3}+u^{-5})\}$$
$$u^2 = xy,\ \theta = \cosh^{-1} u \qquad (36),$$

the threshold frequencies being given by

$$x \cdot y = 1. \qquad (36\,\mathrm{a})$$

It will be noticed that the effect depends on the product of x and y, which is in accordance with the LORENTZ invariance of this product.

[1] BREIT and WHEELER, Bull. American Phys. Soc. 9, 34, (1934). In a communication to the writer the authors have pointed out that their formula quoted in this publication must be divided by 2 on account of a slip in the calculations, and by another factor of 2 on account of their different definition of cross section. The cross section, σ, used here is the more usual one and is such that if in one beam N_1 photons cross unit area, and in the other beam travelling in the opposite direction N_2 photons cross unit area, then $N_1 N_2 \sigma$ is the number of pairs produced per unit area perpendicular to the directions of propagation. The cross section σ' used by BREIT and WHEELER is such that if there are n_1 photons per unit volume in one beam and n_2 in the other, then $n_1 n_2 c \sigma'$ is the number of pairs produced per second per unit volume. It can be shown that $\sigma = \frac{1}{2}\sigma'$.

We shall now derive from (36) the formula for pair-production by a high energy photon and a nucleus. Let the energy of the incident photon in the system, S, in which the nucleus is at rest, be ξmc^2. To bring out the correspondence between this effect and the radiative effect we shall consider a system S' in wich the nucleus is moving in the opposite direction to the photon with a velocity v given by $(1 - v^2/c^2)^{-1/2} = \frac{1}{2}\xi$. The transformation to this system reduces the energy of the photon to mc^2.[1] The photon distribution in the virtual radiation field to which the nuclear field in the system S' is equivalent, is, from (8),

$$N(\nu)\,d\nu = (2/\pi)\,\alpha z^2 \log(g\,\xi\,mc^2/h\nu)\,d\nu/\nu, \quad g \sim 1. \quad (28)$$

The "nuclear" photons travel in the opposite direction to the "external" photon, mc^2, and the cross-section for the production of a pair in terms of formula (36) for the production of pairs by 2 photons, is

$$\sigma(\xi) = \int_{mc^2/h}^{\sim \xi mc^2/h} N(\nu)\,d\nu \quad \sigma(1,\ h\nu/mc^2). \quad (37)$$

Using the Breit-Wheeler formula for $\sigma(x, y)$ this gives on integration

$$\sigma(\xi) = (28/9)\,\alpha z^2\,(e^2/mc^2)^2 \log(g\,\xi). \quad (38)$$

Apart from the actual value of the numerical factor g inside the logarithmic term, which cannot be determined by the present method, this formula is identical with the formula for pair-production by high-energy photons derived by direct

[1] Strictly speaking to $mc^2 \div \left\{1 + \left(1 - \frac{4}{\xi^2}\right)^{\frac{1}{2}}\right\}$, but since we assume $\xi >> 1$ we can take this, for simplicity, as mc^2. The use of the exact value would require a corresponding change in the lower limit to the integral in (37), leaving the result unchanged.

application of DIRAC's theory by HEITLER and SAUTER[1], and by NISHINA, TOMONAGA and SAKATA[2].

(38) is of exactly the same form as the radiative formula (34). The correspondence between the two effects, according to the present method of calculation, arises from the correspondence between the KLEIN-NISHINA scattering formula and the BREIT-WHEELER pair-production formula. For frequencies large compared with mc^2/h the KLEIN-NISHINA scattering cross section is of the order of

$$(e^2/mc^2)^2\,(mc^2/h\nu)\log(h\nu/mc^2) \tag{39}$$

and this also represents the order of magnitude of the cross section (36) for pair-production by a photon $h\nu >> mc^2$ and a photon $h\nu = mc^2$, travelling in opposite directions. Using this approximate form, and substituting x for $h\nu/mc^2$ (37) becomes

$$\sigma(\xi) \approx (e^2/mc^2)^2\,\alpha\, z^2 \int_1^{\sim\xi} x^{-2}\log x \log(g\,\xi/x)\,dx\,. \tag{40}$$

The integral in this expression is exactly the same as that which occurs in the expression (33) for the radiative effect, and its rapid convergence shows that, as in the radiative effect, it is only frequencies in the nuclear field of the order of mc^2/h that are important.

The convergence of the integral in (33) and (40) is, in fact, essential to the full applicability of the present method. In the system S' the nucleus moves with a velocity v given by $(1-v^2/c^2)^{-1/2} \sim \xi$, and the replacement of its field by the radiation represented by (28) is valid only if the per-

[1] W. HEITLER and F. SAUTER, Nature 132, 892, (1934).

[2] Y. NISHINA, S. TOMONAGA and S. SAKATA, Scientific Papers Japanese Inst. Phys. and Chem. Res., 24, 1, (1934).

turbed particles have velocities u satisfying $(1-u^2/c^2)^{-1/2} << (1-v^2/c^2)^{-1/2}$. If this is not so then the fact that the nuclear field in S', is not travelling with exactly the velocity of light can, as it were, be found out by the perturbed particles. Now since in the system S' the pair production is due to the interaction of photons of energy of the order of mc^2, the energy of the positive and negative electrons produced is also of this magnitude. In other words if we denote by u the order of magnitude of the velocity of the electrons concerned in the phenomenon (whether in the negative energy state before interaction, or in a state of positive energy after) then $(1-u^2/c^2)^{-1/2}$ is of the order of unity, and therefore the above condition is satisfied since $(1-v^2/c^2)^{-1/2} \sim \xi >> 1$. Similarly in the radiative effect the velocity u of the electron in S' (acquired through the scattering of the 'nuclear' radiation), is such that $(1-u^2/c^2)^{-1/2} \sim 1$, because the frequencies concerned in the scattering are of the order of only mc^2/h.

Finally we shall refer to the effect of shielding and to the dimensions of the region around the nucleus from which pairs are produced. This region depends upon the distance from the nucleus at which frequencies sufficiently high to satisfy the threshold condition, (36 a), for pair production, are present in the nuclear field. Now in the system S' in which the "external" quantum has energy mc^2 the threshold frequency for the "nuclear" radiation is also mc^2. Frequencies of this order are present to an appreciable extent in the field of an unshielded nucleus out to a distance p_{max} given by $p_{max}/\xi \sim \hbar/mc$ so that $p_{max} \sim \xi(\hbar/mc)$. As a result the region for pair production by a photon $h\nu = \xi mc^2$ extends to a distance $\xi(\hbar/mc)$ from the nucleus.

If the distance $\xi(\hbar/mc)$ is comparable with or greater than atomic dimensions then the shielding of the nuclear field by the atomic electrons will evidently reduce the cross section for pair-production. Assuming the shielding effect to be a sudden cutting off of the nuclear field at $r = z^{-1/3}\cdot$hydrogen radius, then if $\xi > \alpha^{-1} z^{-1/3}, = 137 z^{-1/3}$, the virtual radiation of the nucleus is given by (9), and the pair production formula is obtained by replacing ξ in the log term in (38) by $137 \cdot z^{-1/3}$.

§ 9. Production of Electron-pairs in collisions between two particles, with relative velocity v satisfying $(1 - v/c) \ll 1$.

The cross-section for this effect may be derived by the present method by replacing the field of one of the particles by radiation and then consider the production of pairs by this virtual radiation in the field of the other particle, using the formula given in the previous section. It may also be derived by replacing the fields of both particles with radiation and starting with the Breit-Wheeler formula, but after one integration equivalent to that made in the previous section (to derive the effect of a photon and a nucleus from the effect of two photons) this procedure is identical with the first.

Let v be the relative velocity of the two particles A_1 and A_2, and let $(1 - v^2/c^2)^{-1/2} = \xi$. In a system S in which A_1 is at rest the distribution of photons which represents the field of A_2 (charge $z_2 e$) is

$$N(\nu)\,d\nu = (2/\pi)\,\alpha\, z_2^2 \log(g\,\xi\, mc^2/h\nu)\,d\nu/\nu. \qquad (28)$$

The cross-section $\sigma_p(E)\,dE$ for the creation of a pair of total energy between E and $E + dE$ (including energy of mass) is now the product of $N(\nu)\,d\nu$, with $\nu = E/h$, and the cross section for the production of a pair by a photon of energy E in the field of the stationary particle A_1 (charge $z_1 e$). Substituting for the latter the value given by (38) we accordingly have

$$\left.\begin{aligned}\sigma_p(E)\,dE &= (2/\pi)\,\alpha\xi_2^2\log(g\xi mc^2/E)\,dE/E\cdot\\ &\quad\cdot(28/9)\,\alpha z_1^2(e^2/mc^2)^2\log(g'E/mc^2)\\ &= (56/9\pi)\,\alpha^2 z_1^2 z_2^2(e^2/mc^2)^2\log(g\xi mc^2/E)\\ &\quad\log(0.14\,E/mc^2)\,dE/E.\end{aligned}\right\}\quad(41)^1$$

The maximum frequency in the virtual radiation of A_2 is of the order $\xi mc^2/h$, so that the cross-section for the production of a pair of any energy is $\int_{2mc^2}^{\sim\xi mc^2}\sigma_p(E)\,dE$. On integration this gives

$$\sigma_p(\xi) = (28/27\pi)\,\alpha^2 z_1^2 z_2^2(e^2/mc^2)^2(\log g\xi)^3. \qquad (42)^2$$

With regard to this integrated cross-section it is important that its value is little dependent on the exact value of the differential cross-section $\sigma_p(E)$ near the limits $E_{min}\sim mc^2$ and $E_{max}\sim\xi mc^2$. By virtue of this it is only the numerical coefficient g inside the log term that is affected by the considerations, first that the formula used for pair-production by a photon and a nucleus is correct only in order of magnitude for $E\sim mc^2$, and secondly that for $E\sim\xi mc^2$ the velocities of the corresponding positive and negative electrons do not satisfy the condition for the accurate replacement of the field of A_2 by radiation.

[1] In the second expression we have substituted for g' the exact numerical value given by the calculations of Heitler and Sauter.

[2] The form of this expression is in agreement with calculations of pair-production by two particles carried out by Landau and Lifshitz (Nature, 134, 109 (1934)), in so far as their results are published.

(42) is also valid if one or both of the particles are electrons[1]. This is so because the effect of the reaction on the electrons in a pair-producing collision is negligible unless the energy E of the pairs produced is of the order of ξmc^2. As mentioned in the preceding paragraph pairs of this energy do not make a significant contribution to the integrated cross-section, and a first order change in their number only affects the numerical coefficient g inside the logarithmic term. For an electron of energy ξmc^2, and a nucleus charge ze, we can therefore still use (42). We must of course substitute unity for z_2, and take z_1 equal to the atomic number z, so that the cross section is

$$\sigma_{electron} = (28/27\,\pi)\,\alpha^2 z^2 (e^2/mc^2)^2 (\log g\,\xi)^3. \qquad (43)$$

The shielding of the nucleus by the atomic electrons comes in, in this case, through its effects on the formula for pair-production by a photon and a nucleus used in the derivation of (41). It follows from the results of the preceding section that the differential cross-section given by (41) is unaffected by shielding if E is less than about $137\,mc^2 z^{-1/3}$, while if E is much greater than this then in the second logarithmic term in (41) $(0.14 E/mc^2)$ must be replaced by $(137\,z^{-1/3})$.

§ 10. Splitting of a photon in the field of an electric particle.

In addition to producing a pair of positive and negative electrons the interaction of two photons with electrons in negative energy states may result in a scattering process,

[1] The value of the numerical coefficient g in the logarithmic term will, however, be slightly different for electrons than for heavier particles.

without pair-production. An interaction of this kind between a photon and a virtual photon in the field of an electric particle would result in the latter being scattered from the field of the particle. The process may be compared with the scattering of the virtual radiation in a radiative collision (§ 7), the main difference being that here the scattering is performed by an electron in a negative energy state in the presence of another photon.

An encounter of a photon with an electric particle may thus result in the production of another photon. In the system in which the electric particle is at rest the two photons which are present after the collision must of course have a resultant energy equal to that of the incident photon. The effect would therefore appear as a splitting of the incident photon in the field of the electric particle. While the present method of considering the problem clearly indicates the possibility of this effect, its application to a quantitative treatment is not as straightforward as in the problems considered in previous sections. This is so because the scattering of two photons involves the formation of virtual pairs of all possible energies, and some of these energies may violate the conditions of applicability given in § 1.

Let us, however, consider a photon $h\nu = \xi mc^2$, $\xi >> 1$, incident on a stationary nucleus, ze. In a system, S', moving with the photon with a velocity v given by $(1 - v^2/c^2)^{-1/2} = \frac{1}{2}\xi$ the energy of the photon is mc^2, while the nuclear field is equivalent to a distribution of photons of frequency ranging from zero to ξmc^2, the number per unit frequency range according to (7) being of the order $\alpha z^2 \nu^{-1} \log\left(\frac{\xi mc^2}{h\nu}\right)$. The theoretical cross-section for the scattering of two photons has not been completely worked out, but recent calculations by HEISENBERG[1] show that for a photon mc^2 and a photon $h\nu$, $< mc^2$, moving in opposite directions, it is of the order of $\alpha^2 (e^2/mc^2)^2 (h\nu/mc^2)^5$. The rapid decrease with decreasing $h\nu$ means that the contribution to the cross-section for "splitting" from the virtual photons with $h\nu << mc^2$ is negligible. The contribution from the virtual photons with $h\nu \sim mc^2$ is of the order of

$$\begin{aligned} \sigma_{splitting} &\sim \alpha^2 (e^2/mc^2)^2 \alpha z^2 \log \xi \\ &\sim \alpha^3 z^2 (e^2/mc^2) \log \xi . \end{aligned} \tag{47}$$

HEISENBERG does not actually give the scattering cross-section for photons with $h\nu >> mc^2$, but it is unlikely that the effect of such photons in the virtual radiation of the nucleus makes any significant addition to (47). It will be noticed that the cross-section given by (47) is about α^{-2}, $= 137^2$, times less than the cross section for pair-production by the same photon.

Summary.

The conditions for replacing the field of an electric particle in collision problems by a radiation field, are considered, and a general formula for the spectrum of the equivalent radiation is given. By using the appropriate form of the spectrum formula several effects produced by electric particles are readily deduced from results in radiation theory. The phenomena considered in this way include the ionisation of atoms by electric particles, nuclear disintegration by fast electrons, radiative collisions, production of electron pairs in the field of an atomic nucleus by high energy photons and electrons, and the splitting of a photon in a nuclear

[1] W. HEISENBERG, Zeit. Phys. (1934).

field. The method of treatment provides a new way of regarding these problems and in some cases it shows that the theoretical basis of the existing formulae is very simple.

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